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On the Structure of Surface Waves

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INTRODUCTION

There is considerable confusion in the engineering and physics literature with regard to the existence and properties of surface waves; it has even been suggested [3] that the term "surface wave" be deleted from the language entirely! Now, in the general mathematical context of initial boundary value problems for hyperbolic equations, it is altogether clear when surface waves exist. The present paper is devoted to describing the structure of these waves for three such problems (cf. [5, 6]): 1) Maxwell's equations in the half space R^3_+ with a "strange" boundary condition; 2) Maxwell's equations in R^3_+ with a "completely reactive" boundary condition; 3) the equations of elasticity in R^2_+ with the classical condition for a free boundary.

A principal source of the confusion surrounding surface waves is that it has been more or less customary to seek a solution to a particular problem in the form of an Ansatz considered appropriate to a special source or set of initial data. The interpretation of the resulting solution is thus Ansatz-dependent: in one form the solution may seem to contain surface waves with certain properties, while these properties or even the surface wave itself may not be evident in another form [2]. In the context of the general initial boundary value problem in R^n_+ with arbitrary finite-energy sources or initial data, the representation of the solution in our approach involves construct-

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ing the resolvent kernel of the spatial part of the operator. This reduces to the solution of a system of linear equations with determinant $D(\xi, \zeta)$, $\xi \in R^{n-1}$, $\zeta \in C$. If now D has real roots $\zeta = k(\xi)$, then surface modes exist, and these roots give their frequencies. A surface mode is a generalized eigenfunction of the spatial part operator with eigenvalue $k(\xi)$. Superposition of these modes on ξ then gives a surface wave corresponding to the root $k(\xi)$.

Representations of the solutions of problems 1)–3) above have been given in [5, 6]. From these representations, we here extract descriptions of the structure of surface waves. This structure in all three cases is splendidly simple. First of all, although the system of equations in 1) and 2) is 6×6, and that in 3) is 5×5, the structure of the surface waves as vector fields on $R_+^3 \times R$, or $R_+^2 \times R$ is essentially trivial: in 1) and 3) they are simply scalar functions times constant vectors. Second, contrary to popular belief, a surface wave need not, in general, decay exponentially in the direction normal to the boundary. Third, data producing pure surface waves (with no accompanying radiation field) cannot have compact support; this property has previously been noted in connection with the launching of such waves [1]. Intuitively, this is eminently reasonable: functions producing pure surface waves are orthogonal to functions producing purely radiating waves (superpositions of reflected plane-wave modes); if such a function had compact support it would simply have no way to reach the surface, and its chances of becoming a surface wave would thus be slim indeed. Finally, the parts of the solutions to problems 1)–3) corresponding to surface waves propagate according to scalar wave equations; the wave equations are one-dimensional in the case of 1) and 3), and two-dimensional in the case of 2). Thus, on the set of initial data producing pure surface waves the 6×6 and 5×5 systems collapse into scalar wave equations.

Knowledge of the structure of surface waves is certainly prerequisite to the design of devices for launching them, and the latter is severely limited if this structure is known only for very special sources. Our results for the cases 1)–3) above explicitly describe the structure of the surface-wave component of the resulting field for any finite-energy source or initial field configuration, and should provide broader possibilities in the design of launching devices in these particular situations.

As previously mentioned, the work [5, 6] forms the basis for studying the structure of the surface waves in problems 1)–3) above, and these are considered successively in Sections 1–3. The results of [5, 6] are quoted here in a more long-winded manner than necessary in order to display the role of the "paramutation relations." It has recently been discovered that such relations are of paramount importance in the study of more complex problems in layered media [7]; the problems 1)–3) may be considered the simplest possible examples of such problems. As an application of the paramutation relations, at the ends of Sections 1 and 2 we prove coercivity theorems which are of interest in their own right.

1. Electromagnetic Waves in R^3_+ with a Strange Boundary Condition

The classical energy-preserving boundary condition for Maxwell's equations—that the tangential component of the electric field vanish on the boundary—does not admit surface waves [5]. It was found in [5] that the strange boundary conditions discovered in [4] do admit such waves, and in this section we consider the structure of the surface waves for one such boundary condition. The simplicity of the structure of the surface wave for this boundary condition is especially evident, and we therefore discuss this example in some detail, although the strange boundary conditions are apparently of less physical interest. At the end of the section we prove a coerciveness theorem which demonstrates that the failure of the strange boundary conditions to be coercive [4] is due entirely to the data producing pure surface waves.

We consider elements $\kappa \in C^6$ as column vectors and write them as pairs of three-vectors $\kappa^1 = {}^t(\kappa_1, \kappa_2, \kappa_3)$, $\kappa^2 = {}^t(\kappa_4, \kappa_5, \kappa_6)$. If $f(x, t) = (f^1, f^2)(x, t)$ is a function of $(x, t) \in R^3 \times R$ with values in C^6, Maxwell's equations for the electromagnetic field in a homogeneous, isotropic, nonconducting medium can be written in the form

$$i\partial_t f(x, t) = E^{-1}A(D)f(x, t) = \Lambda(D)f(x, t), \tag{1.1}$$

where

$$\Lambda = E^{-1}A, \qquad A(D) = \sum_{j=1}^{3} A_j D_j = \begin{pmatrix} 0 & i \cdot \mathrm{rot} \\ -i \cdot \mathrm{rot} & 0 \end{pmatrix},$$
$$E = \mathrm{diag}(\varepsilon I_3, \mu I_3), \tag{1.2}$$

$D_j = -i\partial_j = -i\partial/\partial x_j$, and ε and μ are the electromagnetic parameters of the medium. Here $f^1 \equiv \underline{E}$ and $f^2 \equiv \underline{H}$ are, respectively, the electric and magnetic fields.

The symbol $\Lambda(\eta)$, $\eta = (\xi_1, \xi_2, \rho) \equiv (\xi, \rho) = |\eta|\omega \in R^3$, $\omega \in S^2$, is selfadjoint with respect to the E inner product, $(\alpha, \beta) = {}^{\bar t}\alpha E\beta$, in C^6 (${}^{\bar t}M$ denotes the transposed conjugate of the matrix M); its eigenvalues are

$$\lambda_0(\eta) \equiv 0, \qquad \lambda_j(\eta) = jc|\eta|, \qquad c = (\varepsilon\mu)^{-1/2}, \qquad j = \pm 1, \tag{1.3}$$

each of multiplicity two; and the corresponding orthogonal orthoprojectors in C^6 are

$$P_0(\omega) = \begin{pmatrix} \omega \otimes \omega & 0 \\ 0 & \omega \otimes \omega \end{pmatrix}, \quad P_j(\omega) = 2^{-1}\begin{pmatrix} -\omega \wedge \omega \wedge & -j\mu c\omega \wedge \\ j \in c\omega \wedge & \omega \wedge \omega \wedge \end{pmatrix},$$

$$\omega \times \omega = {}^t\omega\omega = \begin{pmatrix} \omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 \end{pmatrix}, \quad \omega \wedge = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix},$$

$$\omega \wedge \omega \wedge = (\omega \wedge)^2, \quad j = \pm 1, \tag{1.4}$$

with the properties

$$\begin{aligned} \Lambda(\eta) P_k(\omega) &= \lambda_k(\eta) P_k(\omega), \qquad k = 0, \pm 1, \\ {}^t[EP_k(\omega)] &= EP_k(\omega), \\ I &= P_0(\omega) + P_1(\omega) + P_{-1}(\omega). \end{aligned} \tag{1.5}$$

In $R_+^3 = \{x \in R^3 : x_3 > 0\}$, the operator $\Lambda(D)$ of (1.2) with strange boundary condition

$$\begin{aligned} 0 = B_s f(x', 0, t) &= {}^t(f_1(x', 0, t), f_4(x', 0, t)) \\ &\equiv {}^t(E_1(x', 0, t), H_1(x', 0, t)) \end{aligned} \tag{1.6}$$

is formally symmetric in the space $\mathring{\mathcal{H}}$ of functions in $L_2(R_+^3; C^6)$ with the E inner product

$$(f, g) = \int_{\mathbf{R}_+^3} {}^t\bar{f}(x) E g(x)\, dx.$$

Its graph closure on the set $\{f \in \mathcal{D}(\overline{R_+^3}; C^6) : B_s f(x', 0) = 0\}$ is a selfadjoint operator in $\mathring{\mathcal{H}}$ which we denote by $\mathring{\Lambda}_s$; here and below $\mathcal{D}(\overline{R_+^n})$ denotes the space of smooth functions with bounded support in $\overline{R_+^n}$, and $\mathcal{D}(R_+^n)$ the space of smooth functions with compact support in R_+^n. The unitary group $\mathring{\mathcal{H}}_s$ generated by $\mathring{\Lambda}_s$ we denote by $\mathring{U}_s(t) = \exp(-it\mathring{\Lambda}_s)$. The initial boundary value problem (1.1), (1.6), $f_0(x) = f(x, 0) \in \mathcal{D}(\mathring{\Lambda}_s)$ is then solved by $\mathring{U}_s(t) f_0(x)$.

The set $S = \{(\operatorname{grad} \phi, \operatorname{grad} \psi) : \phi, \psi \in \mathcal{D}(R_+^3)\}$ is dense in the null space $\mathcal{N}(\mathring{\Lambda}_s)$ of $\mathring{\Lambda}$, and $f \in \mathring{\mathcal{H}}$ is thus in the orthogonal complement $\mathcal{H} = \mathring{\mathcal{H}} \ominus \mathcal{N}(\mathring{\Lambda}_s)$ of $\mathring{\Lambda}$ in $\mathring{\mathcal{H}}$ if and only if $\operatorname{div} f^1 = \operatorname{div} f^2 = 0$ in $\mathcal{D}'(R_+^3)$. The parts of $\mathring{\Lambda}_s$ and $\mathring{U}_s(t)$ in $\mathcal{H}$ we denote by Λ_s and $U_s(t)$.

The group of $U_s(t)$ in $\mathcal{H}$ has a representation as a superposition of two types of generalized eigenfunctions of Λ_s—reflected planewave modes and surface modes—which are defined, respectively, by ($j = \pm 1$)

$$\begin{aligned}\Psi_j(x,\eta) &= (2\pi)^{-3/2}\chi_{-j}(\rho)e^{ix'\xi}\left[e^{ix_3\rho}I - e^{-ix_3\rho}C_s(\eta)\right]P_j(\eta)E^{-1}\\ &= (2\pi)^{-3/2}\chi_{-j}(\rho)e^{ix'\xi}\left[e^{ix_3\rho}P_j(\eta) - e^{-ix_3\rho}P_j(\tilde\eta)C_s(\eta)\right]E^{-1}\end{aligned} \tag{1.7}$$

$$\Sigma_j(x,\eta) = 2(2\pi)^{-3/2}\exp(ix'\xi - |\xi_2|x_3)|\xi_2|e_j(\xi_2)\,{}^t\bar{e}_j(\xi_2)(\rho - i|\xi_2|)^{-1}, \tag{1.8}$$

where

$$C_s(\eta) = \left(\xi_2^2 + \rho^2\right)^{-1}\begin{bmatrix} C(\eta) & 0 \\ 0 & C(\eta)\end{bmatrix},$$

$$C(\eta) = \begin{bmatrix} \rho^2 + \xi_2^2 & 0 & 0 \\ 0 & \xi_2^2 - \rho^2 & 2\rho\xi_2 \\ 0 & -2\rho\xi_2 & \xi_2^2 - \rho^2 \end{bmatrix}$$

$$e_j(\xi_2) = 2^{-1}\varepsilon^{1/2}|\xi_2|^{-1}{}^t\left(0, \varepsilon^{-1}|\xi_2|, i\varepsilon^{-1}\xi_2, 0, -jic\xi_2, jc|\xi_2|\right), \tag{1.9}$$

$$\tilde\eta = (\xi, -\rho),$$

and $\chi_{-j}(\rho)$ is the characteristic function of $R_{-j} = \{\rho \in R : -j\rho > 0\}$.

The matrices Ψ_j, Σ_j satisfy the "eigenvalue" equations (cf. (1.3))

$$\begin{aligned}\Lambda(D)\Psi_j(x,\eta) &= jc|\eta|\Psi_j(x,\eta),\\ \Lambda(D)\Sigma_j(x,\eta) &= jc\xi_1\Sigma_j(x,\eta),\end{aligned} \tag{1.10}$$

and the columns of $\Psi_j(x', x_3, \eta)$ and $\Sigma_j(x', x_3, \eta)$ satisfy the boundary condition (1.5) and are divergence free. The functions $jc\xi_1$ here are the surface-wave frequencies.

The second form of (1.7) is obtained from the first via the reflection paramutation relation

$$C(\eta)P_j(\eta) = P_j(\tilde\eta)C(\eta), \tag{1.11}$$

and the reflection coefficient $C(\eta)$, furthermore, has the properties

$${}^tC(\eta) = C(\tilde\eta), \qquad {}^tC(\eta)C(\eta) = I. \tag{1.12}$$

We denote by $\Phi_n f$ the Fourier transform of a function $f \in L_2(R^n, C^6)$,

$$\Phi_n f(\eta) = (2\pi)^{-n/2}\int_{\mathbf{R}^n}\exp(-ix\eta)f(x)\,dx;$$

the inverse transform is then $\Phi_n^* f(p) = \Phi_n f(-p)$. If $\mathring{\mathcal{H}}$ denotes the set $L_2(R^3, C^6)$ with the E inner product

$$\langle f, f \rangle = \|f\|^2 = \int_{R^3} {}^{t}\bar{f} E f,$$

then (see (1.4))

$$\mathcal{H} = \Phi_3^*(I - P_0)\Phi_3 \mathring{\mathcal{H}} \equiv \Phi_3^*(P_1 + P_{-1})\Phi_3 \mathring{\mathcal{H}} \tag{1.13}$$

is the space of divergence-free, square-integrable functions on R^3 with the E inner product.

The functions (1.7), (1.8) define linear mappings of norm one

$$\begin{aligned} &\Psi_j : \mathcal{H} \to \mathcal{H}, \qquad \Psi_j f(\eta) = \int_{R^3_+} {}^{t}\bar{\Psi}(x, \eta) E f(x)\, dx, \\ &\Sigma_j : \mathcal{H} \to \mathcal{H}, \qquad \Sigma_j f(\eta) = \int_{R^3_+} {}^{t}\bar{\Sigma}_j(x, \eta) E f(x)\, dx, \end{aligned} \tag{1.14}$$

with adjoints

$$\begin{aligned} &\Psi_j^* : \mathcal{H} \to \mathcal{H}, \qquad \Psi_j^* g(x) = \int_{R^3} \Psi_j(x, \eta) E g(\eta)\, d\eta, \\ &\Sigma_j^* : \mathcal{H} \to \mathcal{H}, \qquad \Sigma_j^* g(x) = \int_{R^3} \Sigma_j(x, \eta) E g(\eta)\, d\eta. \end{aligned} \tag{1.15}$$

These mappings are first defined on compactly supported data in $\mathcal{H}$ and $\mathcal{H}$ and then extended to all of $\mathcal{H}$ and $\mathcal{H}$ by the Parseval equality. In general, the integrals are thus understood to be convergent in the topology of $\mathcal{H}$ or $\mathcal{H}$.

Denoting by $(\cdot)_1 : R^3 \to R$ the function $(\cdot)_1(\eta) = \xi_1$, the principal result of [5] concerning Λ_s, $U_s(t)$ can now be stated as follows.

THEOREM 1.1. *The four operators* $\Pi_j = \Psi_j^* \Psi_j$, $\Pi_j^\sigma = \Sigma_j^* \Sigma_j$, $j = \pm 1$, *are all mutually orthogonal orthoprojectors in* $\mathcal{H}$ *which reduce* $U_s(t)$, *and in terms of the maps* (1.14) *and* (1.15), *the latter is represented by*

$$\begin{aligned} U_s(t) f &= \sum_{j = \pm 1} \left[\Psi_j^* \exp(-ijc|\cdot|t) \Psi_k f + \Sigma_j^* \exp(-ijc(\cdot)_1 t) \Sigma_j f \right] \\ &= \sum_{j = \pm 1} \int_{R^3} \left[\Psi_j^*(x, \eta) \exp(-ijc|\eta|t) \Psi_k f(\eta) \right. \\ &\quad \left. + \Sigma_j^*(x, \eta) \exp(-ijc\xi_1 t) \Sigma_j f(\eta) \right] d\eta \end{aligned} \tag{1.16}$$

which, in particular, for $t = 0$ gives the Parseval identity for $f \in \mathcal{H}$:

$$|f|^2 = \sum_{j=\pm 1} \left[\|\Psi_j f\|^2 + \|\Sigma_j f\|^2 \right]. \tag{1.17}$$

Moreover, $f \in \mathfrak{D}(\Lambda_s)$ if and only if

$$c|\eta|\Psi_j f(\eta),\ c\xi_1 \Sigma_j f(\eta) \text{ are in } \mathcal{H}, \tag{1.18}$$

and for such f

$$\begin{aligned} \Lambda_s f &= \sum_{j=\pm 1} \left[\Psi_j^* c|\cdot|\Psi_j f + \Sigma_j^* c(\cdot)_1 \Sigma_j f \right], \\ |\Lambda_s f|^2 &= \sum_{j=\pm 1} \left[\|c|\cdot|\Psi_j f\|^2 + \|c(\cdot)_1 \Sigma_j f\|^2 \right]. \end{aligned} \tag{1.19}$$

From (1.8), (1.14), and (1.15), the explicit formula for the projection $\Pi_j^\sigma = \Sigma_j^* \Sigma_j$ is

$$\begin{aligned} \Pi_j^\sigma f(x) = 2(2\pi)^{-1/2} &\int_{\mathbf{R}} d\xi_2 \exp(ix_2\xi_2 - |\xi_2|x_3)|\xi_2| e_j(\xi_2) {}^t\bar{e}_j(\xi_2) E \\ &\times \int_0^\infty e^{-|\xi|y_3} \Phi_1 f(x_1, \xi_2, y_3)\, dy_3. \end{aligned} \tag{1.20}$$

It is easy to verify directly that Π_j^σ is a projection, i.e., $(\Pi_j^\sigma)^2 = \Pi_j^\sigma$; note that

$$\begin{aligned} &\Phi_1 \Pi_j^\sigma f(x_1, \xi_2, x_3) \\ &\quad = 2e^{-|\xi_2|x_3}|\xi_2| e_j(\xi_2) {}^t\bar{e}_j(\xi_2) E \int_0^\infty e^{-|\xi_2|y_3} \Phi_1 f(x_1, \xi_1, y_3)\, dy_3; \end{aligned}$$

now multiply by ${}^t\bar{e}_j(\xi_2) E \exp(-|\xi_2|x_3)$, and integrate on x_3 over R_+.

For $f \in \mathcal{H}$ we now define (cf. (1.9), (1.20))

$$e_j^\pm = 2^{-1}\varepsilon^{1/2}\,{}^t(0, \varepsilon^{-1}, \pm i\varepsilon^{-1}, 0, \mp jic, jc) \tag{1.21}$$

$$\begin{aligned} l_j^\pm(x_1, \xi; f) = 2(2\pi)^{-1} &\chi_\pm(\xi) {}^t\bar{e}_j^\pm E \\ &\times \int_{R_+^2} \exp[-i\xi(y_2 \mp iy_3)] f(x_1, y_2, y_3)\, dy_2\, dy_3, \end{aligned} \tag{1.22}$$

$$F_j^\pm(x_1, x_2 \pm ix_3) = \int_{R_\pm} \exp[i\xi(x_2 \pm ix_3)] |\xi| l_j^\pm(x_1, \xi)\, d\xi. \tag{1.23}$$

Here $\chi_\pm$ is the characteristic function of $R_\pm$.

THEOREM 1.2. *For any $f \in \mathcal{H}$, the projection $\Pi_j^\sigma f$ of f onto $\Pi_j^\sigma \mathcal{H}$ can be represented in terms of the scalar functions* (1.23) *and constant vectors* (1.21) *in the form*

$$\Pi_j^\sigma f(x) = e_j^+ F_j^+(x_1, x_2 + ix_3) + e_j^- F_j^-(x_1, x_2 - ix_3), \qquad (1.24)$$

and the action of $U_s(t)$ on $\Pi_j^\sigma f$ is simply translation

$$\begin{aligned} U_s(t)\Pi_j^\sigma f(x) &= \Pi_j^\sigma f(x_1 - jct, x_2, x_3) \\ &= e_j^+ F_j^+(x_1 - jct, x_2 + ix_3) + e_j^- F_j^-(x_1 - jct, x_2 - ix_3). \end{aligned} \qquad (1.25)$$

In particular, the structure of data $f \in \Pi_j^\sigma \mathcal{H}$ as a vector field is trivial, and no nonzero such data may be compactly supported in R_+^3. If $f \in \mathfrak{D}(\Lambda_s) \cap \Pi_j^\sigma \mathcal{H}$, then the solution of the initial boundary value problem (1.1), (1.5), *with initial data f is given by* (1.25) *without the projection Π_j^σ, and hence on $\Pi_j^\sigma \mathcal{H}$, Maxwell's equations* (1.1) *degenerate into the diagonal, first-order, one-dimensional, scalar wave equation $(\partial_t + jc\partial_{x_1})I_6 f = 0, j = \pm 1$.*

Proof. Formula (1.24) follows from (1.20)–(1.23), and (1.25) is then obtained via (1.16). It is seen from (1.23) that the $F_j^\pm$ are harmonic in R_+^2 for a.e. x_1 and may therefore be compactly supported if and only if they vanish for a.e. x_1.

Remark. It is easily verified that any functions $l_\pm(x_1, \xi) \in L_2(R \times R_\pm, C)$ such that $\xi \to |\xi|^{1/2} D_1 l_\pm(\cdot, \xi) \in L_2(R_\pm, L_2(R))$ define a function $f \in \mathfrak{D}(\Lambda_s) \cap \Pi_j^\sigma \mathcal{H}$ via (1.22), (1.23), and the right hand side of (1.24).

It is straightforward computation to show that for $\phi \in \mathfrak{D}(R)$ the one-parameter family of functions

$$f_\tau(x) = \phi(x_1) e_1^+ \exp(i\tau x_2 - \tau x_3)(x_2 + ix_3 + i)^{-2}, \qquad 0 \leqslant \tau < \infty, \qquad (1.26)$$

is in $\Pi_1^\sigma \mathcal{H}$, i.e., $f_\tau \in \mathcal{H}$ and $\Pi_1^\sigma f_\tau = f_\tau$. For $\tau = 0$ we thus see from (1.25) that

$$U_s(t) f_0(x) = f_0(x_1 - ct, x_2 + ix_3)$$

does not decay exponentially away from the boundary $x_3 = 0$. More generally, we have the

COROLLARY 1.3. *Let f be any smooth function in $\mathfrak{D}(\Lambda_s) \cap \Pi_j^\sigma \mathcal{H}$. There exists $\tau > 0$ such that*

$$|f(x)|_{C^6} \leqslant K(x_1)\exp(-\tau x_3) \qquad (1.27)$$

if and only if $\operatorname{supp} l_+(x_1, \cdot) \subset [\tau, \infty)$ *and* $\operatorname{supp} l_-(x_1, \cdot) \subset (-\infty, -\tau]$ *for a.e.* x_1 (*see* (1.22)–(1.24)).

Proof. Consider, for example, $f = e_j^+ F_j^+(x_1, x_2 + ix_3)$. Taking the Fourier transform on x_2 and multiplying by $\exp(\xi x_3)$ in (1.23) gives

$$\chi_+(\xi) l_+(x_1, \xi) = \chi_+(\xi) \int_R \exp[-i\xi(x_2 + ix_3)] F_j(x_1, x_2 + ix_3)\, dx_2,$$

with integrand analytic in R^2_+ for fixed x_1. If now (1.27) holds, then

$$\left|\exp[-i\xi(x_2 + ix_3)] F_j(x_1, x_2 + ix_3)\right| < K(x_1)\exp(\xi - \tau)x_3,$$

so for $0 < \xi < \tau$ $l_+(x_1, \xi) = 0$ by Cauchy's theorem. Conversely, if $\operatorname{supp} l_+(x_1, \cdot) \subset [\tau, \infty)$, then the estimate (1.27) follows by a simple direct estimate of (1.23).

The family of functions (1.26) is precisely that used in [4] to demonstrate that Λ_s is not coercive on $\mathfrak{D}(\Lambda_s)$. Since $f_\tau \in \Pi^\sigma \mathcal{H} \equiv \Pi_1^\sigma \mathcal{H} \oplus \Pi_{-1}^\sigma \mathcal{H}$, it is natural to ask whether Λ_s is coercive on the smaller subspace $\mathcal{H} \ominus \Pi^\sigma \mathcal{H} = \Pi_1 \oplus \Pi_{-1}\mathcal{H}$, i.e., does Λ_s fail to be coercive precisely because of the existence of surface waves (see the discussion in Section 0 of [5])? The answer is affirmative. We include the proof here as an application of the paramutation relations (1.11), (1.12).

THEOREM 1.4. *Let* $f \in (\Pi_1 + \Pi_{-1})\mathcal{H} \cap \mathfrak{D}(\Lambda_s)$. *Then* $f \in H^1(\mathbf{R}^3_+)$ *and*

$$2|\Lambda_s f|^2 \geqslant c^2 \sum_{j=1}^{3} |D_j f|^2.$$

Proof. The proof consists of a sequence of simple observations some of which are of interest in themselves. For $f \in \mathcal{H}$ from (1.7) and (1.14)

$$\begin{aligned} \Psi_j f(\eta) &= \chi_{-j}(\rho) P_j(\eta) (2\pi)^{-1/2} \int_0^\infty \left[e^{-ix_3\rho} I - C(\tilde{\eta}) e^{ix_3\rho}\right] \Phi_2 f(\xi, x_3)\, dx_3 \\ &\equiv \chi_{-j}(\rho) P_j(\eta) \Phi_3 G(\eta), \end{aligned} \tag{1.28}$$

where

$$\begin{aligned} \Phi_3 G(\eta) &= (2\pi)^{-1/2} \int_{-\infty}^\infty e^{-ix_3\rho} [\chi_+(x_3) \Phi_2 f(\xi, x_3) \\ &\quad - \chi_-(x_3) C(\tilde{\eta}) \Phi_2 f(\xi, -x_3)]\, dx_3 \\ &= (2\pi)^{-1/2} \int_0^\infty \left[e^{-ix_3\rho} - C(\tilde{\eta}) e^{ix_3\rho}\right] \Phi_2 f(\xi, x_3)\, dx_3. \end{aligned} \tag{1.29}$$

We first show that (cf. (1.4), (1.5))

$$P_0(\eta)\Phi_3 G(\eta) = 0, \tag{1.30}$$

so that G is in the complement of the null space of the selfadjoint operator Λ' in $\mathring{\mathcal{H}}$ engendered by $\Lambda(D)$ of (1.3) on the domain

$$\mathcal{D}(\Lambda') = \{f \in \mathring{\mathcal{H}}: \Lambda f \in \mathring{\mathcal{H}}\}, \tag{1.31}$$

i.e., $G \in \mathring{\mathcal{H}} \ominus \mathcal{N}(\Lambda') = \mathcal{H}$. We remark that Λ' is coercive on $\mathcal{D}(\Lambda') \cap \mathcal{H}$: for $F \in \mathcal{D}(\Lambda') \cap \mathcal{H}$

$$\begin{aligned}
\|\Lambda' F\|^2 &= \int_{R^3} |\Lambda(\eta)[P_0(\eta) + P_1(\eta) + P_{-1}(\eta)]\Phi_3 F(\eta)|^2\, d\eta \\
&= \int_{R^3} |\Lambda(\eta)P_1(\eta)\Phi_3 F(\eta) + \Lambda(\eta)P_{-1}(\eta)\Phi_3 F(\eta)|^2\, d\eta \\
&= c^2 \int_{R^3} |\eta|^2 |\Phi_3 F(\eta)|^2 \\
&= c^2 \sum_{j=1}^{3} \|D_j F\|^2.
\end{aligned} \tag{1.32}$$

By Theorem 3.2 of [4] there exists a sequence $\{f_n\} \subset H^1(R^3_+) \cap C(\overline{R^3_+}) \cap \mathring{\mathcal{H}}$ such that $f_n \to f$ in $\mathcal{H}$, and hence $\Phi_3 G_n \to \Phi_3 G$ in $\mathring{\mathcal{H}}$, where G_n is defined by (1.29) with f_n in place of f. For f_n the following integration by parts makes sense:

$$\begin{aligned}
&\int_0^\infty \exp(\pm i x_3 \rho) D_3 \Phi_2 f_n(\xi, x_3)\, dx_3 \\
&\qquad = i\Phi_2 f(\xi, 0) \mp \rho \int_0^\infty \exp(\pm i x_3 \rho) \Phi_2 f_n(\xi, x_3)\, dx_3.
\end{aligned}$$

Now note that from (1.9),

$$\eta C(\tilde{\eta}) = \tilde{\eta} = (\xi_1, \xi_2, -\rho).$$

Hence

$$\begin{aligned}
\eta \Phi G_n^1(\eta) &= \eta \int_0^\infty \exp(-i x_3 \rho) \Phi_2 f_n^1(\xi, x_3)\, dx_3 \\
&\quad - \tilde{\eta} \int_0^\infty \exp(i x_3 \rho) \Phi_2 f_n^1(\xi, x_3)\, dx_3 \\
&= \int_0^\infty \exp(-i x_3 \rho) \Phi_2 \operatorname{div} f_n^1(\xi, x_3)\, dx_3 \\
&\quad - \int_0^\infty \exp(i x_3 \rho) \Phi_2 \operatorname{div} f_n^1(\xi, x_3)\, dx_3 = 0,
\end{aligned}$$

since the f_n are divergence-free. Similarly, $\eta\Phi_3 G_n^2(\eta) = 0$, and so by (1.4) $P_0(\eta)\Phi_3 G_n(\eta) = 0$. Finally, letting $n \to \infty$, $P_0(\eta)\Phi_3 G(\eta) = 0$. Next from (1.29),

$$G(x) = \chi_+ f(x) - \Phi^* C(\tilde{\cdot})\Phi\chi_- f(x), \tag{1.33}$$

and it is easy to show by direct computation from (1.9) and the residue calculus that

$$\chi_+(x_3)\Phi^* C(\tilde{\cdot})\Phi\chi_- f(x) = \Pi^\sigma f(x) \equiv (\Pi_1^\sigma + \Pi_{-1}^\sigma) f(x). \tag{1.34}$$

Finally, from (1.11), (1.12), and (1.28),

$$-C(\eta)\Psi_j f(\eta) = \chi_{-j}(\rho) P_j(\tilde{\eta})\Phi G(\tilde{\eta}). \tag{1.35}$$

Suppose now that $f \in (\Pi_1 + \Pi_{-1})\mathcal{H} \cap \mathfrak{D}(\Lambda_s)$; then by (1.12), (1.18), and (1.35),

$$\begin{aligned}
\infty > \int_{R^3_{-j}} |\eta|^2 |P_j(\eta)\Phi_3 G(\eta)|^2 \, d\eta &= \int_{R^3_{-j}} |\eta|^2 |\Psi_j f(\eta)|^2 \, d\eta \\
&= \int_{R^3_{-j}} |\eta|^2 |C(\eta)\Psi_j f(\eta)|^2 \, d\eta \\
&= \int_{R^3_{-j}} |\eta|^2 |P_j(\tilde{\eta})\Phi_3 G(\tilde{\eta})|^2 \, d\eta \\
&= \int_{R^3_j} |\eta|^2 |P_j(\eta)\Phi_3 G(\eta)|^2 \, d\eta,
\end{aligned}$$

and hence by (1.4), (1.30),

$$\infty > \sum_{j=\pm 1} \int_{R^3} |\eta|^2 P_j \Phi_3 G(\eta)|^2 \, d\eta = 2 \sum_{j=\pm 1} \int_{R^3_{-j}} |\eta|^2 |\Psi_j f(\eta)|^2 \, d\eta. \tag{1.36}$$

Therefore, $G \in \mathfrak{D}(\Lambda') \cap \mathcal{H}$, and thus (1.32) holds. Now by (1.4), (1.17), (1.30), (1.32), (1.34), and (1.36),

$$\begin{aligned}
\infty > 2|\Lambda_s f|^2 &= 2 \sum_{j=\pm 1} \int_{R^3_{-j}} c^2 |\eta|^2 |\Psi_j f(\eta)|^2 \, d\eta \\
&= c^2 \sum_{j=\pm 1} \int_{R^3} |\eta|^2 |P_j(\eta)\Phi_3 G(\eta)|^2 \, d\eta \\
&= c^2 \int_{R^3} |\eta|^2 |\Phi_3 G(\eta)|^2 \, d\eta
\end{aligned}$$

$$= c^2 \sum_{j=1}^{3} \|D_j G\|^2$$

$$\geqslant c^2 \sum_{j=1}^{3} \int_{R^3_+} |D_j G(x)|^2 \, dx$$

$$= c^2 \sum_{j=1}^{3} \int_{R^3_+} |D_j[f(x) - \Phi_3^* C(\check{\cdot})\Phi\chi_- f(x)]|^2 \, dx$$

$$= c^2 \sum_{j=1}^{3} \int_{R^3_+} |D_j f(x)|^2 \, dx$$

$$= c^2 \sum_{j=1}^{3} |D_j f|^2,$$

by (1.34) and the fact that $\Pi^\sigma f = 0$.

2. Electromagnetic Waves in R^3_+ with a Completely Reactive Boundary Condition

In the case, for example, of two semi-infinite media with distinct electromagnetic properties separated by a common plane boundary, the so-called reactive or Leontovich boundary conditions are often used to approximately describe the reflected field in one of the media, due to radiation incident from a source in this same medium. The so-called "completely reactive" boundary condition

$$B_r f(x', 0, t) = 0, \; B_r = \begin{pmatrix} 1 & 0 & 0 & 0 & i\alpha & 0 \\ 0 & 1 & 0 & -i\alpha & 0 & 0 \end{pmatrix}, \qquad 0 < \alpha \in R, \quad (2.1)$$

is energy-preserving for Maxwell's equations, i.e., if (2.1) is satisfied, then

$$(\Lambda f, f) = i\int_{R^2} {}^t\bar{f}(x',0) A_3 f(x',0) \, dx' + (f, \Lambda f) = (f, \Lambda f).$$

The graph closure of Λ on $\{f \in \mathcal{D}(\overline{R^3_+}, C^6) : B_r f(x', 0) = 0\}$ is a selfadjoint operator $\mathring{\Lambda}_r$ which generates the unitary group $\mathring{U}_r(t) = \exp(-i\mathring{\Lambda}_r t)$ in $\mathring{\mathcal{H}}$. It is easily seen that the set of functions $\{(\nabla\phi, \nabla\psi) : \phi, \psi \in \mathcal{D}(R^3_+)\}$ is dense in $\mathfrak{N}(\mathring{\Lambda}_r)$, the null space of $\mathring{\Lambda}_r$ (cf. Remark 4.2 of [4]). Data are thus in the complement of the null space of $\mathring{\Lambda}_r$ in $\mathring{\mathcal{H}}$ if and only if they are divergence-free. We set $\mathcal{H} = \mathring{\mathcal{H}} \ominus \mathfrak{N}(\mathring{\Lambda}_r)$ and denote the parts of $\mathring{\Lambda}_r$, $\mathring{U}_r(t)$ in $\mathcal{H}$ by

Λ_r, $U_r(t)$. The derivation of the representation of $U_r(t)$ in $\mathcal{H}$ in terms of reflected plane waves and surface waves is step-for-step the same as in [5]. We therefore present only the results.

The boundary condition (2.1) admits two types of surface modes: E (or TE) modes in which the electric field is transverse to the plane of incidence, and M (or TM) modes, in which the magnetic field is so directed. The generalized eigenfunctions of Λ_r corresponding to the reflected plane-wave modes and the E and M surface modes are, respectively, ($j = \pm 1$)

$$\begin{aligned}\Psi_j(x,\eta) &= (2\pi)^{-3/2}\chi_{-j}(\rho)\left[e^{ix\eta} - C_j^r(\omega)e^{ix\tilde{\eta}}\right]P_j(\omega)E^{-1}\\ &= (2\pi)^{-3/2}\chi_{-j}(\rho)\left[e^{ix\eta}P_j(\omega) - e^{ix\tilde{\eta}}P_j(\tilde{\omega})C_j^r(\omega)\right]E^{-1}, \quad (2.2)\end{aligned}$$

$$\begin{aligned}\Sigma_E(x,\eta) &= (2\pi)^{-3/2}q\mu(c\alpha\varepsilon)^{-1}|\xi|^{-3}\exp(ix'\xi - \tau_E x_3)e_E{}^t\bar{e}_E(\rho + i\tau_E)^{-1},\\ \Sigma_M(x,\eta) &= (2\pi)^{-3/2}p\alpha\varepsilon|\xi|^{-3}\exp(ix'\xi - \tau_M x_3)e_M{}^t\bar{e}_M(\rho + i\tau_M)^{-1}, \quad (2.3)\end{aligned}$$

where

$$\begin{aligned}C_j^r(\omega) &= j\varepsilon c\omega_3(\mu\varepsilon^{-1} + \alpha^2)\Delta_j^{-1}Q + i\alpha\Delta_j^{-1}C_r(\omega'),\\ Q &= \operatorname{diag}(-1,-1,1,1,1,-1),\\ C_r(\omega') &= \begin{pmatrix} C(\omega') & 0\\ 0 & C(\omega')\end{pmatrix}.\\ C(\omega') &= \begin{pmatrix}\omega_2^2 - \omega_1^2 & -2\omega_1\omega_2 & 0\\ -2\omega_1\omega_2 & \omega_1^2 - \omega_2^2 & 0\\ 0 & 0 & \omega_1^2 + \omega_2^2\end{pmatrix},\\ \Delta_j(\omega) &= (\mu c - ij\alpha\omega_3)(i\alpha\varepsilon c - j\omega_3) = i\alpha(1 + \omega_3^2) + j\varepsilon c\omega_3(\alpha^2 - \mu\varepsilon^{-1}),\end{aligned}$$

and

$$\begin{aligned}e_E(\xi) &= {}^t(k_E\xi_2, -k_E\xi_1, 0, i\mu^{-1}\tau_E\xi_1, i\mu^{-1}\tau_E\xi_2, -\mu^{-1}|\xi|^2),\\ e_M(\xi) &= {}^t(\varepsilon^{-1}\tau_M\xi_1, \varepsilon^{-1}\tau_M\xi_2, i\varepsilon^{-1}|\xi|^2, ik_M\xi_2, -ik_M\xi_1, 0),\\ k_E(\xi) &= cq|\xi|,\ q^2 = \alpha^2(\alpha^2 + \mu\varepsilon^{-1})^{-1},\\ \tau_E(\xi) &= \alpha^{-1}\mu k_E(\xi) = \alpha^{-1}\mu cq|\xi|,\\ k_M(\xi) &= -\varepsilon^{-1}p|\xi|,\ p^2 = (\mu\varepsilon^{-1} + \alpha^2)^{-1},\\ \tau_M(\xi) &= -\alpha\varepsilon k_M(\xi) = \alpha p|\xi|.\end{aligned} \quad (2.5)$$

The second form of $\Psi_j(x, \eta)$ follows from the first via the reflection paramutation relation (cf. (1.11))

$$C_j^r(\omega)P_j(\omega) = P_j(\tilde{\omega})C_j^r(\omega). \tag{2.6}$$

The reflection coefficient $C_j^r(\omega)$ again has the properties (cf. (1.12))

$$\begin{aligned} {}^t\bar{C}_j^r(\omega) &= C_j^r(\tilde{\omega}), \qquad \tilde{\omega} = (\omega_1, \omega_2, -\omega_3), \\ {}^t\bar{C}_j^r(\omega)C_j^r(\omega) &= I. \end{aligned} \tag{2.7}$$

The functions (2.2) and (2.3) satisfy the "eigenvalue" equations

$$\begin{aligned} \Lambda(D)\Psi_j(x, \eta) &= jc|\eta|\Psi_j(x, \eta), \\ \Lambda(D)\Sigma_E(x, \eta) &= k_E(\xi)\Sigma_E(x, \eta), \\ \Lambda(D)\Sigma_M(x, \eta) &= k_M(\xi)\Sigma_M(x, \eta), \end{aligned}$$

while the columns of $\Psi_j(x, \eta)$, $\Sigma_E(x, \eta)$, and $\Sigma_M(x, \eta)$ satisfy the boundary conditions (2.1) and are divergence-free.

As in (1.14), (1.15) the mappings of norm one

$$\begin{aligned} &\Psi_j : \mathcal{H} \to \mathcal{H}, \qquad \Psi_j f(\eta) = \int_{R^3_+} {}^t\bar{\Psi}_j(x, \eta)Ef(x)\,dx, \\ &\Sigma_S : \mathcal{H} \to \mathcal{H}, \qquad \Sigma_S f(\eta) = \int_{R^3_+} {}^t\bar{\Sigma}_S(x, \eta)Ef(x)\,dx, \qquad S = E, M, \end{aligned} \tag{2.8}$$

have adjoints

$$\begin{aligned} &\Psi_j^* : \mathcal{H} \to \mathcal{H}, \qquad \Psi_j^* g(x) = \int_{R^3} \Psi_j(x, \eta)Eg(\eta)\,d\eta, \\ &\Sigma_S^* : \mathcal{H} \to \mathcal{H}, \qquad \Sigma_S^* g(x) = \int_{R^3} \Sigma_S(x, \eta)Eg(\eta)\,d\eta, \qquad S = E, M. \end{aligned} \tag{2.9}$$

The analogue of Theorem 1.1 is now

THEOREM 2.1. *The four operators* $\Pi_j = \Psi_j^*\Psi_j$, $j = \pm 1$, $\Pi_S = \Sigma_S^*\Sigma_S$, $S = E, M$ *are all mutually orthogonal orthoprojectors in* $\mathcal{H}$ *which reduce* $U_r(t)$, *and in terms of the maps* (2.8) *and* (2.9), *the latter is respresented by*

$$U_r(t)f = \sum_{j=\pm 1} \Psi_j^* \exp(-ijc|\cdot|t)\Psi_j f + \sum_{S=E, M} \Sigma_S^* \exp[-ik_S(\cdot)t]\Sigma_S f \tag{2.10}$$

which, in particular, for $t = 0$ gives the Parseval identity for $f \in \mathcal{H}$

$$f = \sum_{j=\pm 1} \Psi_j^* \Psi_j f + \sum_{S=E,M} \Sigma_S^* \Sigma_S f. \tag{2.11}$$

Moreover, $f \in \mathcal{D}(\Lambda_r)$ if and only if

$$c|\eta|\Psi_j f, \qquad k_S(\xi)\Sigma_S f(\eta), \qquad j = \pm 1, S = E, M, \text{ are in } \mathcal{H} \tag{2.12}$$

and for such f

$$\Lambda_r f = \sum_{j=\pm 1} \Psi_j^* jc| \cdot |\Psi_j f + \sum_{S=E,M} \Sigma_s^* k_S(\cdot)\Sigma_S f,$$

$$|\Lambda_r f|^2 = \sum_{j=\pm 1} \|c| \cdot |\Psi_j f\|^2 + \sum_{S=E,M} \|k_S(\cdot)\Sigma_S f\|^2. \tag{2.13}$$

Remark. The fact that the projections are orthogonal follows from the fact that (see [5]) $0 = \Psi_{-1}\Psi_1^* = \Psi_{-1}\Sigma_S^* = \Psi_1\Sigma_S^*$, $S = E$, M, $\Sigma_E \Sigma_M^* = 0$, and the relations obtained by taking adjoints of these.

From (2.3), (2.8), and (2.9), the projections $\Pi_S = \Sigma_S^* \Sigma_S$, $S = E, M$, can easily be computed explicitly. They are obtained by setting $t = 0$ in the formula (2.14) below. From (2.10) we thus obtain

THEOREM 2.2. *For any initial data $f \in \mathcal{H}$, the orthogonal surface-wave components of $U_r(t)f$ are*

$$\begin{aligned} U_r(t)\Pi_E f(x) &= (2\pi)^{-1} q\mu(c\alpha\varepsilon)^{-1} \int_{R^2} \exp[ix'\xi - \tau_E x_3 - ik_E t] \\ &\quad \times |\xi|^{-3} e_E{}^t e_E E \int_0^\infty e^{-\tau_E(\xi) y_3} \Phi_2 f(\xi, y_3)\, dy_3 \\ &= (2\pi)^{-1} q\mu(c\alpha\varepsilon)^{-1} \int_{R^2} \exp\left[ix'\xi - \alpha^{-1}\mu c q|\xi| - icq|\xi| t\right] \\ &\quad \times |\xi|^{-3} e_E{}^t e_E E \int_0^\infty \exp\left(-\alpha^{-1}\mu c q|\xi| y_3\right) \Phi_2 f(\xi, y_3)\, dy_3, \end{aligned} \tag{2.14}$$

$$\begin{aligned} U_r(t)\Pi_M f(x) &= (2\pi)^{-1} \alpha\varepsilon p \int_{R^2} \exp[ix'\xi - \tau_M x_3 - ik_M t] |\xi|^{-3} e_M{}^t e_M E \\ &\quad \times \int_0^\infty e^{-\tau_M(\xi) y_3} \Phi_2 f(\xi, y_3)\, dy_3 \\ &= (2\pi)^{-1} \alpha\varepsilon p \int_{R^2} \exp\left[ix'\xi - \alpha p|\xi| x_3 + i\varepsilon^{-1} p|\xi| t\right] \\ &\quad \times |\xi|^{-3} e_M{}^t e_M E \int_0^\infty e^{-\alpha p|\xi| y_3} \Phi_2 f(\xi, y_3)\, dy_3, \end{aligned} \tag{2.15}$$

and these waves (formally) satisfy the two-dimensional, scalar wave equations

$$
\begin{aligned}
&\left[\partial_t^2 - c^2q^2\left(\partial_1^2 + \partial_2^2\right)\right] I_6 \cdot U_r(t)\Pi_E f = 0,\\
&\left[\partial_t^2 - \varepsilon^{-2}p^2\left(\partial_1^2 + \partial_2^2\right)\right] I_6 \cdot U_r(t)\Pi_M f = 0.
\end{aligned}
\tag{2.16}
$$

If $f = \Pi_S f$, $S = E, M$, and f has compact support in R^3_+, then $f \equiv 0$.

The last assertion follows from the fact that if, say, $f = \Pi_M f$, then $h(x) = f(x_1, x_2, (\alpha p)^{-1}x_3)$ is harmonic. Just as in Section 1, it is easy to verify directly that Π_E and Π_M are projections.

We observe that the structure of the surface-wave components as vector fields is not entirely trivial; we proceed to do something about this alarming state of affairs. We define

$$
\begin{aligned}
\tilde{e}_E(\xi) &= c^{-1}q^{-1}|\xi|^{-1}e_E(\xi)\\
&= {}^t\!\left(\xi_2, -\xi_1, 0, i\alpha^{-1}\xi_1, i\alpha^{-1}\xi_2, -\mu^{-1}c^{-1}q^{-1}|\xi|\right)\\
\tilde{e}_E(D) &= {}^t\!\left(D_2, -D_1, 0, i\alpha^{-1}D_1, i\alpha^{-1}D_2, i\alpha\mu^{-1}\varepsilon q^{-2}D_3\right)\\
\tilde{e}_M(\xi) &= \varepsilon p^{-1}|\xi|^{-1}e_M(\xi)\\
&= {}^t\!\left(\alpha\xi_1, \alpha\xi_2, ip^{-1}|\xi|, -i\xi_2, i\xi_1, 0\right),\\
\tilde{e}_M(D) &= {}^t\!\left(\alpha D_1, \alpha D_2, \alpha^{-1}p^{-2}D_3, -iD_2, iD_1, 0\right),
\end{aligned}
\tag{2.17}
$$

so that

$$
\begin{aligned}
\tilde{e}_E(\xi) &= \tilde{e}_E(D)\exp\left(ix'\xi - \alpha^{-1}\mu cq|\xi|x_3\right),\\
\tilde{e}_M(\xi) &= \tilde{e}_M(D)\exp\left(ix'\xi - \alpha p|\xi|x_3\right).
\end{aligned}
\tag{2.18}
$$

From (2.14) and (2.15) with $t = 0$, the projections can now be written

$$
\begin{aligned}
\Pi_E f(x) = (2\pi)^{-1}\mu cq^3(\alpha\varepsilon)^{-1}\tilde{e}_E(D)\int_{R^2} d\xi \exp\left(ix'\xi - \alpha^{-1}\mu cq|\xi|x_3\right)|\xi|^{-1}\\
\times\left\{{}^t\bar{e}_E(\xi)E\int_0^\infty dy_3 \exp\left(-\alpha^{-1}\mu cq|\xi|y_3\right)\Phi_2 f(\xi, y_3)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\Pi_M f(x) = (2\pi)^{-1}\alpha\varepsilon^{-1}p^3\tilde{e}_M(D)\int_{R^2} d\xi \exp\left(ix'\xi - \alpha p|\xi|x_3\right)|\xi|^{-1}\\
\times\left\{{}^t\bar{\tilde{e}}_M(\xi)E\int_0^\infty dy_3 \exp\left(-\alpha p|\xi|y_3\right)\Phi_2 f(\xi, y_3)\right\}
\end{aligned}
\tag{2.19}
$$

THEOREM 2.3. *For any $f \in \mathcal{H}$ the orthogonal surface-wave components of $U_r(t)f$ have the form*

$$\begin{aligned} U_r(t)\Pi_E f(x) &= (2\pi)^{-1}\tilde{e}_E(D) \\ &\quad \times \int_{R^2} d\xi \exp\left(ix'\xi - \alpha^{-1}\mu c q|\xi|x_3 - ik_E t\right)|\xi|^{-1} l_E(\xi, f), \\ U_r(t)\Pi_M f(x) &= (2\pi)^{-1}\tilde{e}_M(D) \\ &\quad \times \int_{R^2} d\xi \exp(ix'\xi - \alpha p|\xi|x_3 - ik_M t)|\xi|^{-1} l_M(\xi, f), \end{aligned} \tag{2.20}$$

where the scalar functions $|\xi|^{1/2} l_E(\xi, f)$, $|\xi|^{1/2} l_M(\xi, f)$ are in $L_2(R^2)$ (these are the terms in braces in (2.19)). *Conversely, if $|\xi|^{1/2} l(\xi) \in L_2(R^2)$ and the functions $L_E(x)$, $L_M(x)$ are defined by*

$$\begin{aligned} L_E(x) &= (2\pi)^{-1}\int_{R^2} \exp\left(ix'\xi - \alpha^{-1}\mu c q|\xi|x_3\right) l(\xi)\, d\xi, \\ L_M(x) &= (2\pi)^{-1}\int_{R^2} \exp(ix'\xi - \alpha p|\xi|x_3) l(\xi)\, d\xi \end{aligned} \tag{2.21}$$

(*note that* $l = \Phi_2 L_E(\cdot, 0) = \Phi_2 L_M(\cdot, 0)$)*, then* $\tilde{e}_E(D)L_E \in \Pi_E\mathcal{H}$ *and* $\tilde{e}_M(D)L_M \in \Pi_M\mathcal{H}$. $U_r(t)\tilde{e}_E(D)L_E(x)$ *and* $U_r(t)\tilde{e}_M(D)L_M(x)$ *are given by the right sides of* (2.20) *with* $l_{E,M} \equiv l$.

Proof. The first assertion is immediate from (2.14), (2.15), and (2.19). If $|\xi|^{1/2} l(\xi) \in L_2(R^2)$, then from (2.21)

$$\int_0^\infty |\Phi_2 \tilde{e}_S(D)_i L_S(\xi, x_3)|^2\, dx_3 \leqslant \text{const} \cdot |\xi| \cdot |l(\xi)|^2,$$
$$S = E, M, \quad i = 1, \ldots, 6,$$

so $\tilde{e}_S(D)L_S$ are in $L_2(R^3_+)$, and since $\operatorname{div} \tilde{e}_S(D)^1 L_S(x) = \operatorname{div} \tilde{e}^2_S(D)^2 L_S(x) = 0$, it follows that $\tilde{e}_S(D)L_S$ are in $\mathcal{H}$, $S = E, M$. Now

$$(2\pi)^{-1}\int_{R^3_+} \exp\left(-iy'\xi - \alpha^{-1}\mu c q|\xi|y_3\right)\tilde{e}_E(D)L_E(y)\, dy$$
$$= \beta\Phi_2 L_E(\xi, 0) - \gamma\int_{R^3_+} \exp\left(-iy'\xi - \alpha^{-1}\mu c q|\xi|y_3\right) L_E(y)\, dy,$$

where

$$\beta = {}^t(0, 0, 0, 0, 0, -\alpha\mu^{-1}\varepsilon q^{-2}),$$
$$\gamma = {}^t\left(-\xi_2, \xi_1, 0, -i\alpha^{-1}\xi_1, -i\alpha^{-1}\xi_2, -\mu^{-1}c^{-1}q^{-1}|\xi|\right),$$

and

$$ {}^t\tilde{e}_E(\xi) E\beta = \alpha\mu^{-1}c^{-1}\varepsilon q^{-3}|\xi|, $$
$$ {}^t\tilde{e}_E(\xi) E\gamma = 0. $$

Hence from (2.19)

$$ \Pi_E \tilde{e}_E(D) L_E(x) = \tilde{e}_E(D) L_E(x), $$

and so $\tilde{e}_E(D)L_E \in \Pi_E\mathcal{K}$. The computation to verify that $\tilde{e}_M(D)L_M \in \Pi_M\mathcal{K}$ is the same.

COROLLARY 2.4. *There exist smooth data g_E, g_M in $\Pi_E\mathcal{K}, \Pi_M\mathcal{K}$ such that*

$$ |\Phi_2 g_E(\xi, x_3)| \leqslant |e(\xi)| \exp(-\tau x_3), $$
$$ |\Phi_2 g_M(\xi, x_3)| \leqslant |m(\xi)| \exp(-\tau x_3), \qquad 0 < \tau \in \mathbb{R}, $$

with $e, m \in L_2(R^2)$ if and only if the supports of $l_E(\xi, g)$ and $l_M(\xi, g)$ are contained, respectively, in $\{|\xi| > \alpha\mu^{-1}c^{-1}q^{-1}\tau\}$ and $\{|\xi| > \alpha^{-1}p^{-1}\tau\}$. Thus, data in $\Pi_{E,M}\mathcal{K}$ need not decay exponentially away from $x_3 = 0$.

Proof. It suffices to consider, for example, L_E of (2.21). Suppose that $|\Phi_2 L_E(\xi, x_3)| \leqslant |e(\xi)| \exp(-\tau x_3)$. From (2.21)

$$ \Phi_2 L_E(\xi, x_3) = \exp\left(-\alpha^{-1}\mu c q |\xi| x_3\right) l(\xi), $$

and so

$$ |l(\xi)| \leqslant |e(\xi)| \exp\left[\left(\alpha^{-1}\mu c q|\xi| - \tau\right)x_3\right] $$

letting $x_3 \to \infty$ gives $l(\xi) = 0$ if $|\xi| < \alpha\mu^{-1}c^{-1}q^{-1}\tau$. Conversely, if $\operatorname{supp} l_E \subset \{|\xi| > \alpha\mu^{-1}c^{-1}q^{-1}\tau\}$, then

$$ \begin{aligned} |\Phi_2 L_E(\xi, x_3)| &= \chi_{\{|\xi| > \alpha\mu^{-1}c^{-1}q^{-1}\tau\}}(\xi) \cdot \left| l(\xi) \exp\left(-\alpha^{-1}\mu c q |\xi| x_3\right)\right| \\ &\leqslant \exp(-\tau x_3) |l(\xi)|. \end{aligned} $$

Finally, to illustrate the application of the reflection paramutation relation (2.6), (2.7), we prove that Λ_r is coercive on $\mathfrak{D}(\Lambda_r)$ in $\mathcal{K}$.

THEOREM 2.5. *Let $d = \min\{\alpha^{-2}\varepsilon^{-2}, \alpha^2, c^2q^2, \varepsilon^{-2}p^2\}$, and $a = \min\{d/3, c^2/2\}$. For $f \in \mathfrak{D}(D\Lambda_r)$*

$$ |\Lambda_r f|^2 \geqslant a \sum_{l=1}^{3} |D_l f|^2. $$

Remark. It was noted in [5] that if a boundary condition admits a surface mode with frequency $k(\xi)$ that vanishes for nonzero $|\xi|$ (such as the frequencies $\pm c\xi_1$ for B_s in Section 1), then the operator with this boundary condition is noncoercive. The present theorem is an example of the converse assertion: the frequencies $k_E(\xi)$ and $k_M(\xi)$ do not vanish for nonzero $|\xi|$.

Proof of the theorem. For $f \in \mathcal{H}$ we set ($\tilde{\omega} = (\omega_1, \omega_2, -\omega_3)$)

$$\hat{f}_j(\eta) = (2\pi)^{-3/2} P_j(\omega) \int_{R^3_+} \left[e^{-ix_3\rho} I - C_j^r(\tilde{\omega}) e^{ix_3\rho}\right] \Phi_2 f(\xi, x_3)\, dx_3.$$

Then from (2.2), (2.8)

$$\Psi_j f(\eta) = \chi_{-j}(\rho) \hat{f}_j(\eta).$$

Now (2.6) and (2.7) imply that for $\eta = (\xi, \rho) \in R^3$

$$C_j^r(\tilde{\omega}) \hat{f}_j(\tilde{\eta}) = -\hat{f}_j(\eta), \qquad \tilde{\eta} = (\xi, -\rho), \tag{2.22}$$

and hence from (2.9)

$$\begin{aligned} \Psi_j^* \Psi_j f(x) &= (2\pi)^{-3/2} \int_{R^3_{-j}} \left[e^{ix\eta} I - e^{ix\tilde{\eta}} C_j^r(\omega)\right] \hat{f}_j(\eta)\, d\eta \\ &= (2\pi)^{-3/2} \left\{ \int_{R^3_{-j}} e^{ix\eta} \hat{f}_j(\eta)\, d\eta - \int_{R^3_{+j}} e^{ix\eta} C_j^r(\tilde{\omega}) \hat{f}_j(\tilde{\eta})\, d\eta \right\} \\ &= \Phi_3^* \hat{f}_j(x), \end{aligned} \tag{2.23}$$

and also, again using (2.7) and (2.22)

$$\|\Psi_j f\|^2 = \int_{R^3_{-j}} |\hat{f}_j(\eta)|^2\, d\eta = \int_{R^3_{-j}} |C_j^r(\tilde{\omega}) \hat{f}_j(\tilde{\eta})|^2\, d\eta = \int_{R^3_{+j}} |\hat{f}_j(\eta)|^2\, d\eta,$$

so that

$$\|\hat{f}_j\|^2 = \int_{R^3} |\hat{f}_j(\eta)|^2\, d\eta = 2\int_{R^3_{-j}} |\hat{f}_j(\eta)|^2\, d\eta = 2\|\Psi_j f\|^2. \tag{2.24}$$

Hence, if $f \in \mathfrak{D}(\Lambda_r)$, then by (2.13), (2.23), and (2.24) with the notation $(\cdot)_l(\eta) = \eta_l$, we have for $l = 1, 2, 3$,

$$\begin{aligned} \infty > 2\|c| \cdot |\Psi_j f\|^2 &= c^2 \||\cdot|\hat{f}_j\|^2 \\ &= c^2 \sum_{l=1}^{3} \|(\cdot)_l \hat{f}_j\|^2 \\ &= c^2 \sum_{l=1}^{3} \|\Phi_3^* (\cdot)_l \hat{f}_j\|^2 \end{aligned}$$

$$= c^2 \sum_{l=1}^{3} \|D_l \Phi_3^* \hat{f}_j\|^2$$

$$\geqslant c^2 \sum_{l=1}^{3} |D_l \Psi_j^* \Psi_j f|^2,$$

and hence

$$\infty \geqslant 2 \sum_{j=\pm 1} \|c| \cdot |\Psi_j f\|^2 \geqslant c^2 \sum_{l=1}^{3} \sum_{j=\pm 1} |D_l \Psi_j^* \Psi_j f|^2$$
$$\geqslant c^2 \sum_{l=1}^{3} \left| D_l \sum_{j=\pm 1} \Psi_j^* \Psi_j f \right|^2. \tag{2.25}$$

From (2.3), (2.5), (2.9), (2.12) and the fact that Σ_S^*, $S = E, M$, has norm one, we have for $k = 1, 2$ and $f \in \mathcal{D}(\Lambda_r)$,

$$\alpha^{-2} \varepsilon^{-2} |D_3 \Sigma_M^* \Sigma_M f|^2 \leqslant \|k_m \Sigma_M f\|^2 \geqslant c^2 q^2 |D_k \Sigma_M^* \Sigma_M f|^2,$$
$$\alpha^2 |D_3 \Sigma_E^* \Sigma_E f|^2 \leqslant \|k_E \Sigma_E f\|^2 \geqslant \varepsilon^{-2} p^2 |D_k \Sigma_E^* \Sigma_E f|^2,$$

and hence

$$3 \sum_{S=E,M} \|k_S \Sigma_S f\|^2 \geqslant d \sum_{S=E,m} \sum_{l=1}^{3} \left| D_l \Sigma_S^* \Sigma_S f \right|^2$$
$$\geqslant d \sum_{l=1}^{3} \left| D_l \sum_{S=E,M} \Sigma_S^* \Sigma_S f \right|^2. \tag{2.26}$$

Hence from (2.11), (2.13), (2.25), and (2.26) for $f \in \mathcal{D}(\Lambda_r)$

$$|\Lambda_r f|^2 = \sum_{j=\pm 1} \|c| \cdot |\Psi_j f\|^2 + \sum_{S=E,M} \|k_S \Sigma_S f\|^2$$
$$\geqslant a \sum_{l=1}^{3} \left\{ \left| D_l \sum_{j=\pm 1} \Psi_j^* \Psi_j f \right|^2 + \left| D_l \sum_{s=E,M} \Sigma_S^* \Sigma_S f \right|^2 \right.$$
$$\geqslant a \sum_{l=1}^{3} \left| D_l \left\{ \sum_{j=\pm 1} \Psi_j^* \Psi_j f + \sum_{S=E,M} \Sigma_S^* \Sigma_S f \right\} \right|^2$$
$$= a \sum_{l=1}^{3} |D_l f|^2.$$

3. Elastic Waves in R_+^2

The equations of elasticity are much tougher to deal with than Maxwell's equations, because they admit two nonzero propagation speeds and, in general, the corresponding modes become coupled at the boundary (see [6]). Nevertheless, with sufficient perseverance it is possible to derive a simple, explicit expression for the surface-wave components—the Rayleigh waves—of any finite-energy elastic disturbance propagating in a homogeneous, isotropic elastic medium filling $R^2_+ = \{x = (x_1, x_2) \in R : x_2 > 0\}$ with a free boundary $x_2 = 0$. As in Section 1, these components propagate according to the one-dimensional scalar wave equation, and their structure as vector fields is trivial. The work [6] is the basis for our discussion, but we have written things in a somewhat different form in order to display the paramutation relations. The principal result is stated in Theorem 3.4.

Denoting by σ_{ij} the components of the stress tensor, and by $v = (v_1, v_2)$ the displacement velocity, in terms of the vector-valued function on $R^2 \times R$ $f(x, t) = {}^t(\sigma_{11}, \sigma_{22}, \sigma_{12}, v_1, v_2)(x, t)$ the equations of elasticity in two space dimensions are

$$-i\partial_t f(x, g) = \mathcal{A}(D) f(x, t), \tag{3.1}$$

where

$$\begin{gathered}
\mathcal{A}(D) = E^{-1}A(D) \\
A(D) = \begin{pmatrix} 0_{3x3} & A_0(D) \\ {}^tA_0(D) & 0_{2x2} \end{pmatrix}, \qquad {}^tA_0(D) = \begin{pmatrix} D_1 & 0 & D_2 \\ 0 & D_2 & D_1 \end{pmatrix}, \\
E = \begin{pmatrix} E_0 & 0 \\ 0 & I_2 \end{pmatrix}, \qquad E_0^{-1} = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, \\
E_0 = [4\mu(\lambda + \mu)]^{-1} \begin{pmatrix} \lambda + 2\mu & -\lambda & 0 \\ -\lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 4(\lambda + \mu) \end{pmatrix},
\end{gathered} \tag{3.2}$$

and λ, μ are the Lamé parameters of the medium. The symbol $\mathcal{A}(\eta)$, $\eta = (\xi, \rho) = |\eta|\omega \in R^2$ is selfadjoint in C^5 with the E inner product, $(\alpha, \beta) = {}^t\alpha E \beta$, and has the five orthonormal eigenvectors and corresponding eigenvalues ($k = \pm 1$)

$$\begin{aligned}
&\lambda_0(\omega) \equiv 0, \qquad e_0(\omega) = 2(ab)^{1/2}\,{}^t\big(\omega_2^2, \omega_1^2, -\omega_1\omega_2, 0, 0\big), \\
&\lambda_{ks}(\omega) = k\mu^{1/2}, \\
&e_{ks}(\omega) = 2^{-1/2}\,{}^t\big(2\mu^{1/2}\omega_1\omega_2, -2\mu^{1/2}\omega_1\omega_2, \mu^{1/2}\big(\omega_2^2 - \omega_1^2\big), k\omega_2, -k\omega_1\big), \\
&\lambda_{kp}(\omega) = kc^{-1}, \\
&e_{kp}(\omega) = 2^{-1/2}\,{}^t\big(c\lambda + 2b\omega_1^2, c\lambda + 2b\omega_2^2, 2b\omega_1\omega_2, k\omega_1, k\omega_2\big), \\
&\qquad c = (\lambda + 2\mu)^{-1/2}, \qquad a = (\lambda + \mu)c, \qquad b = \mu c.
\end{aligned} \tag{3.3}$$

From (3.2) and (3.3)

$$Ee_{ks}(\omega) = (2\mu)^{-1/2}{}^{t}\left(\omega_1\omega_2, -\omega_1\omega_2, \omega_2^2 - \omega_1^2, \mu^{1/2}k\omega_2, -\mu^{1/2}k\omega_1\right),$$
$$Ee_{kp}(\omega) = 2^{-1/2}{}^{t}\left(c\omega_1^2, c\omega_2^2, 2c\omega_1\omega_2, k\omega_1, k\omega_2\right), \tag{3.4}$$

and the orthogonal projections onto the s- and p-subspaces which are selfadjoint with respect to the E inner product in C^5 are

$$P_{ks}(\omega) = e_{ks}(\omega)^t\left[Ee_{ks}(\omega)\right], \qquad P_{kp}(\omega) = e_{kp}(\omega)^t\left[E_{kp}e(\omega)\right]. \tag{3.5}$$

The classical initial boundary value problem in R_+^2 consists in finding a solution of (3.1) subject to the initial and boundary conditions

$$\begin{aligned} f(x,0) &= f_0(x), \\ Bf(x_1,0,t) &\equiv {}^t\left(f_2(x_1,0,t), f_3(x_1,0,t)\right) = 0, \end{aligned} \tag{3.6}$$

i.e., the normal components of the stress tensor σ_{12} and σ_{22} vanish on the boundary. The initial value $f_0 \in \mathring{\mathcal{H}}$, which is the set of functions $f, g \in L_2(R_+^2, C^5)$ with the E inner product

$$(f,g) = \int_{R_+^2} {}^t\bar{f}(x)Eg(x)\,dx. \tag{3.8}$$

The closure in graph norm of the operator $\mathcal{Q}(D)$ on the set $\{f \in \mathfrak{D}(\overline{R_+^2}, C^5) : Bf(x_1,0) = 0\}$ is a selfadjoint operator $\mathring{\mathcal{Q}}$ in $\mathcal{H}$, and the unitary operator $\mathring{U}(t) = \exp(i\mathring{\mathcal{Q}}t)$ it generates delivers the solution $\mathring{U}(t)f_0$ to (3.1), (3.6), (3.7) for initial data f_0 in the domain of $\mathring{\mathcal{Q}}$. Initial data (3.6) giving rise to purely propagating solutions of (3.1) lie in the complement of the null space of $\mathring{\mathcal{Q}}$ in $\mathring{\mathcal{H}}$,

$$\mathcal{H} = \mathring{\mathcal{H}} \ominus \mathfrak{N}((\mathring{\mathcal{Q}})), \tag{3.9}$$

and are characterized by the condition: $f \in \mathcal{H}$ if and only if in $\mathfrak{D}'(R_+^2)$

$$0 = \operatorname{div}_0 f \equiv \left(\partial_2^2 - \lambda c\partial_1^2, \partial_1^2 - \lambda c\partial_2^2, -4ac\partial_1\partial_2, 0, 0\right)f \tag{3.10}$$

(see [6], Theorem 2.1). The parts of $\mathring{U}(t)$ and $\mathring{\mathcal{Q}}$ in $\mathcal{H}$ are denoted by $U(t)$ and $\mathcal{Q}$.

The representation of $U(t)$ in $\mathcal{H}$ consists of superpositions of two types of generalized eigenfunctions for $\mathcal{Q}$: those corresponding to reflected P (pressure) and S (shear) waves, and those corresponding to Rayleigh waves (surface waves). If α is the angle incident S and P waves make with the normal $(0,1)$ to the boundary $x_2 = 0$, then these waves propagate in the direction $\omega = (\omega_1, \omega_2) = (\sin\alpha, -\cos\alpha)$; both give rise to reflected S and P waves propagating in the direction $\tilde{\omega} = (\omega_1, -\omega_2)$ and, respectively, to P and S waves propagating in directions θ and ϕ, where

$$\begin{aligned}
&\phi = (\phi_1, \phi_2) = \left(n\omega_1, |1 - n^2\omega_1^2|^{1/2}\right) = (\sin\alpha_s, \cos\alpha_s),\\
&\theta = (\theta_1, \theta_2) = \left(\sin\alpha_p, \cos\alpha_p\right), \qquad \theta_1 = n^{-1}\omega_1,\\
&\theta_2 = \begin{Bmatrix} |1 - \theta_1^2|^{1/2}, & |\omega_1| < n \\ i|1 - \theta_1^2|^{1/2}, & |\omega_1| > n \end{Bmatrix},\\
&n = \mu^{1/2}c.
\end{aligned} \tag{3.11}$$

The S and P generalized eigenfunctions of $\mathcal{Q}$ are $(k = \pm 1)$

$$\begin{aligned}
\Psi_{ks}(x, \eta) &= \chi_{-k}(\rho)(2\pi)^{-1}e^{ix_1\xi}\Big[e^{i\rho x_2}I - r_{ss}(\omega)C_{ss}e^{-i\rho x_2}\\
&\qquad - r_{sp}(\omega)E^{-1}C_{sp}(\omega)Ee^{\mathrm{in}|\eta|\theta_2 x_2}\Big]P_{ks}(\omega)E^{-1}\\
&= \chi_{-k}(\rho)(2\pi)^{-1}e^{ix_1\xi}\Big[e^{i\rho x_2}I - r_{ss}(\omega)e^{-i\rho x_2}P_{ks}(\tilde{\omega})C_{ss}\\
&\qquad - r_{sp}(\omega)e^{\mathrm{in}|\eta|\theta_2 x_2}P_{kp}(\theta)D'_{ps}(\omega)\Big]E^{-1}\\
&= \chi_{-k}(\rho)(2\pi)^{-1}e^{ix_1\xi}\Big[e^{i\rho x_2}e_{ks}(\omega) - r_{ss}(\omega)e^{-i\rho x_2}e_{ks}(\tilde{\omega})\\
&\qquad - r_{sp}(\omega)n^{-1}\omega_1\omega_2e^{\mathrm{in}|\eta|\theta_2 x_2}e_{kp}(\theta)\Big]^t e_{ks}(\omega)
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
\Psi_{kp}(\omega) &= \chi_{-k}(\rho)(2\pi)^{-1}e^{ix_1\xi}\Big[e^{i\rho x_2}I - r_{pp}(\omega)D_{pp}e^{-i\rho x_2}\\
&\qquad - r_{ps}(\omega)E^{-1}D_{ps}Ee^{\mathrm{in}^{-1}|\eta|\phi_2 x_2}\Big]P_{kp}(\omega)E^{-1}\\
&= \chi_{-k}(\rho)(2\pi)^{-1}e^{ix_1\xi}\Big[e^{i\rho x_2}P_{kp}(\omega) - r_{pp}(\omega)e^{-i\rho x_2}P_{kp}(\tilde{\omega})D_{pp}\\
&\qquad - r_{ps}(\omega)e^{\mathrm{in}^{-1}|\eta|\phi_2 x_2}P_{ks}(\phi)C'_{sp}(\omega)\Big]E^{-1}\\
&= \chi_{-k}(\rho)(2\pi)^{-1}e^{ix_1\xi}\Big[e^{i\rho x_2}e_{kp}(\omega) - r_{pp}(\omega)e^{-i\rho x_2}e_{kp}(\tilde{\omega})\\
&\qquad + n\omega_1\omega_2 r_{ps}(\omega)e^{\mathrm{in}^{-1}|\eta|\phi_2 x_2}e_{ks}(\phi)\Big]^t e_{kp}(\omega),
\end{aligned} \tag{3.13}$$

where

$$\Delta_S(\omega) = \left(2\omega_1^2 - 1\right)^2 - 4n^2\omega_1\omega_2\theta_1\theta_2 = \cos^2\alpha + n^2\sin 2\alpha\sin 2\alpha_p,$$

$$\Delta_P(\omega) = \left(2\phi_1^2 - 1\right)^2 - 4n^2\omega_1\omega_2\phi_1\phi_2 = \cos^2 2\alpha_s + n^2\sin 2\alpha\sin 2\alpha_s,$$

$$\begin{aligned}
&\Delta_S(\omega)r_{ss}(\omega) = \Delta_s(\tilde{\omega}),\\
&\Delta_S(\omega)r_{sp}(\omega) = 4n^2(2\omega_1^2 - 1) = -4n^2\cos 2\alpha,\\
&\Delta_P(\omega)r_{pp}(\omega) = \Delta_P(\tilde{\omega}),\\
&\Delta_P(\omega)r_{ps}(\omega) = -4(1 - 2\phi_1^2) = -4\cos 2\alpha_s,\\
&\qquad C_{ss} = -D_{ss} \equiv Q = \mathrm{diag}(-1, -1, 1, -1, 1),\\
&\qquad C_{sp} = \mathrm{diag}\left(\theta_1^2, -\theta_2^2, 2\omega_1\omega_2\theta_1\theta_2(\omega_2^2 - \omega_1^2)^{-1}, \omega_1^2, -n\omega_2\theta_2\right),\\
&\qquad D_{ps} = \mathrm{diag}\left(-n\omega_2\phi_2, \omega_1\omega_2^{-1}\phi_1\phi_2, \tfrac{1}{2}(\phi_1^2 - \phi_2^2), -n\omega_2\phi_2, \phi_1^2\right),\\
&\qquad C'_{sp} = -n\omega_2\phi_2^{-1}\mathrm{diag}\\
&\qquad\qquad \times\left(\omega_1^2, -\omega_2^2, 2\omega_1\omega_2\phi_1\phi_2(\phi_2^2 - \phi_1^2)^{-1}, \omega_1^2, -n^{-1}\omega_2\phi_2\right),\\
&\qquad D'_{ps} = -n^{-1}\omega_2\theta_2^{-1}\mathrm{diag}\\
&\qquad\qquad \times\left(-n\omega_2\theta_2, \omega_1\omega_2\theta_1\theta_2^{-1}, \tfrac{1}{2}(\omega_1^2 - \omega_2^2), -n\omega_2\theta_2, \omega_1^2\right),
\end{aligned} \tag{3.14}$$

and in (3.12), (3.13)

$$\theta = (\phi_1, k\phi_2), \qquad \theta = (\theta_1, \theta_2^k), \qquad \theta_2^k = \begin{Bmatrix} k\theta_2, & |\omega_1| < n \\ \theta_2 & |\omega_1| > n \end{Bmatrix}.$$

The second equalities in (3.12), (3.13) follow from the paramutation relations which, in particular, relate the operators corresponding to the two different types of waves

$$\begin{aligned}
QP_{ks}(\omega) &= P_{ks}(\tilde{\omega})Q, \qquad QP_{kp}(\omega) = P_{kp}(\tilde{\omega})Q,\\
C_{sp}(\omega)EP_{ks}(\omega) &= EP_{kp}(\theta)D'_{ps}(\omega),\\
D_{ps}(\omega)EP_{kp}(\omega) &= EP_{ks}(\phi)C'_{sp}(\omega).
\end{aligned} \tag{3.15}$$

From this second form it is immediately clear that

$$\begin{aligned}
\mathcal{Q}(D)\Psi_{ks}(x,\eta) &= \lambda_{ks}|\eta|\Psi_{ks}(x,\eta),\\
\mathcal{Q}(D)\Psi_{kp}(x,\eta) &= \lambda_{kp}|\eta|\Psi_{kp}(x,\eta),
\end{aligned} \tag{3.16}$$

since, for example, $\exp(ix_1\xi) = \exp(\mathrm{in}|\eta|\theta_1 x)$, so that

$$\begin{aligned}
&\mathcal{Q}(D)\exp(ix\xi_1 + \mathrm{in}|\eta|\theta_2 x_2)P_{kp}(\theta)\\
&\qquad = n|\eta|\exp(ix_1\xi_1 + \mathrm{in}|\eta|\theta_2 x_2)\mathcal{Q}(\theta)P_{kp}(\theta)\\
&\qquad = \lambda_{kp}n|\eta|\exp(ix_1\xi_1 + \mathrm{in}|\eta|\theta_2 x_2)P_{kp}(\theta)\\
&\qquad = \lambda_{ks}|\eta|\exp(ix_1\xi_1 + \mathrm{in}|\eta|\theta_2 x_2)P_{kp}(\theta).
\end{aligned}$$

The third forms of Ψ_{ks} and Ψ_{kp} given in (3.12), (3.13) are those found in [6]; they follow from the first forms and the relations (cf. (3.15))

$$C_{sp}(\omega)Ee_{ks}(\omega) = n^{-1}\omega_1\omega_2 Ee_{kp}(\theta),$$
$$D_{ps}(\omega)Ee_{kp}(\omega) = -n\omega_1\omega_2 Ee_{ks}(\phi).$$

It is easy to check, using the third form, that

$$B\Psi_{ks}(x_1, 0, \eta) = B\Psi_{kp}(x_1, 0, \eta) = 0. \tag{3.17}$$

Moreover, the columns of Ψ_{ks}, Ψ_{kp} satisfy (3.10), so that superpositions of them give functions in $\mathcal{H}$.

To describe the surface modes we define (see (3.24) for R_0)

$$\begin{aligned} \mathring{\sigma} &= (\mathring{\sigma}_1, \mathring{\sigma}_2) = R_0^{-1}\left(\operatorname{sgn}\xi, i\left(1 - R_0^2\right)^{1/2}\right), \\ \mathring{\pi} &= (\mathring{\pi}_1, \mathring{\pi}_2) = (nR_0)^{-1}\left(\operatorname{sgn}\xi, i\left(1 - n^2R_0^2\right)^{1/2}\right), \\ 1 &= \mathring{\sigma}_1^2 + \mathring{\sigma}_2^2 = \mathring{\pi}_1^2 = \mathring{\pi}_2^2. \end{aligned} \tag{3.18}$$

The surface modes are then ($k = \pm 1$)

$$\Sigma_k(x, \eta) = -i(4\pi)^{-1}\alpha(R_0)^{-1}e^{ix_1\xi}\gamma_k(\xi, x_2)^t\tilde{\gamma}_k(\xi, \rho), \tag{3.19}$$

where

$$\begin{aligned} \gamma_k(\xi, x_2) &= \exp\left[-|\xi|x_2\surd\left(1 - R_0^2\right)\right]\frac{R_0^2 - 2}{\surd\left(1 - R_0^2\right)}e_{ks}(\mathring{\sigma}) \\ &\quad + \exp\left[-|\xi|x_2\surd\left(1 - n^2R_0^2\right)\right]2in\operatorname{sgn}\xi e_{ks}(\mathring{\pi}), \end{aligned} \tag{3.20}$$

$$\tilde{\gamma}_k(\xi, \rho) = \frac{\left(2 - R_0^2\right)|\xi|}{\rho - i|\xi|\surd\left(1 - R_0^2\right)}e_{ks}(\mathring{\sigma}) - \frac{2in\xi\surd\left(1 - R_0^2\right)}{\rho - i|\xi|\surd\left(1 - n^2R_0^2\right)}e_{kp}(\mathring{\pi}),$$

$$\begin{aligned} 0 < \alpha(R_0) &= \left(1 - R_0^2\right)^{-1/2}\left(1 - n^2R_0^2\right)^{-1/2} \\ &\quad \times\left[1 + n^2 - 2n^2R_0^2 - \left(2 - R_0^2\right)^3/4\right]. \end{aligned} \tag{3.21}$$

The surface modes Σ_k satisfy the "eigenvalue equation"

$$\mathcal{Q}(D)\Sigma_k(x, \eta) = R_k(\xi)\Sigma_k(x, \eta), \tag{3.22}$$

where

$$R_k(\xi) = k\mu^{1/2}R_0|\xi|, \qquad 2/3 < R_0 < 1, \tag{3.23}$$

and R_0 is defined by

$$\left(2 - R_0^2\right)^2 = 4\left(1 - R_0^2\right)^{1/2}\left(1 - n^2R_0^2\right)^{1/2}, \qquad n = \mu^{1/2}c. \tag{3.24}$$

They likewise satisfy the boundary condition

$$B\Sigma_k(x_1,0,\eta) = 0, \tag{3.25}$$

and the columns of Σ_k satisfy (3.10).

The functions (3.11), (3.12), and (3.19) define mappings of norm one from $\mathcal{H}$ into $\mathcal{K}$, the set $L_2(R^2, C^5)$ with the E inner product, by

$$\begin{aligned} \Psi_{ks}f(\eta) &= \int_{R^2_+} {}^t\bar{\Psi}_{ks}(x,\eta)Ef(x)\,dx, \\ \Psi_{kp}f(\eta) &= \int_{R^2_+} {}^t\bar{\Psi}_{kp}(x,\eta)Ef(x)\,dx, \\ \Sigma_k f(\eta) &= \int_{R^2_+} {}^t\bar{\Sigma}_k(x,\eta)Ef(x)\,dx, \qquad k = \pm 1, \end{aligned} \tag{3.26}$$

which have adjoints

$$\begin{aligned} \Psi^*_{ks}g(x) &= \int_{R^2} \Psi_{ks}(x,\eta)Eg(\eta)\,d\eta, \\ \Psi^*_{kp}g(x) &= \int_{R^2} \Psi_{kp}(x,\eta)Eg(\eta)\,d\eta, \\ \Sigma^*_k g(x) &= \int_{R^2} \Sigma_k(x,\eta)Eg(\eta)\,d\eta. \end{aligned} \tag{3.27}$$

THEOREM 3.1 [6]. *For $\Delta \subset R$, the spectral measure $F(\Delta)$ for $\mathcal{A}$ in $\mathcal{H}$ has the representation*

$$\begin{aligned} F(\Delta)f = \sum_{k=\pm 1} \Big\{ &\Psi^*_{ks}\chi_{\{\lambda_{ks}|\cdot|\in\Delta\}}\Psi_{ks}f + \Psi^*_{kp}\chi_{\{\lambda_{kp}|\cdot|\in\Delta\}}\Psi_{kp}f \\ &+ \Sigma^*_k\chi_{\{R_k(\cdot)\in\Delta\}}\Sigma_k f\Big\}, \end{aligned} \tag{3.28}$$

and the group $U(t)$ is represented by

$$U(t)f = \sum_{k=\pm 1} \left\{ \Psi^*_{ks}e^{i\lambda_{ks}|\cdot|t}\Psi_{ks}f + \Psi^*_{kp}e^{i\lambda_{kp}|\cdot|t}\Psi_{kp}f + \Sigma^*_k e^{iR_k(\cdot)t}\Sigma_k f \right\}. \tag{3.29}$$

The six operators $\Pi_{ks} \equiv \Psi^*_{ks}\Psi_{ks}$, $\Pi_{kp} \equiv \Psi^*_{kp}\Psi_{kp}$, $\Pi^\sigma_k \equiv \Sigma^*_k\Sigma_k$ are all mutually orthogonal orthoprojectors in $\mathcal{H}$ which reduce $U(t)$. Moreover, $f \in \mathcal{D}(\mathcal{A})$ if and only if $\lambda_{ks}|\eta|\Psi_{ks}f(\eta)$, $\lambda_{kp}|\eta|\Psi_{kp}f(\eta)$, $R_k(\xi)\Sigma_k f(\eta)$ are in $\mathcal{K}$ and for *such* f

$$\mathcal{Q}f = \sum_{k=\pm 1} \{\Psi_{ks}^* \lambda_{ks}| \cdot |\Psi_{ks} f + \Psi_{kp}^* \lambda_{kp}| \cdot |\Psi_{kp} f + \Sigma_k^* R_k(\cdot) \Sigma_k f\}. \tag{3.30}$$

The remainder of the section is devoted to showing that $U(t)$ is simply translation on each of the subspaces $\Pi_k^\sigma \mathcal{H}$. The matter reduces to obtaining a simple representation of $\Pi_k^\sigma f$; along the way we verify directly that Π_k^σ is a projection in $\mathcal{H}$.

From (3.20), (3.26), and (3.27)

$$\begin{aligned}\Pi_k^\sigma f(x) = (4\pi)^{-2} \alpha(R_0)^{-2} &\int_R d\xi\, e^{ix_1\xi} \gamma_k(\xi, x_2) \\ &\times \left(\int_R {}^t\bar{\tilde{\gamma}}_k(\xi, \rho) E \tilde{\gamma}_k(\xi, \rho)\, d\rho \right) \\ &\times \int_{R_+^2} e^{-i\xi y_1}\, {}^t\bar{\gamma}_k(\xi, y_2) E f(y)\, dy. \end{aligned} \tag{3.31}$$

The proof of the first Lemma is straightforward computation using (3.20), (3.24), (3.31), and the residue theorem.

LEMMA 3.2. *Let*

$$\begin{aligned}\tau(R_0) = \sqrt{(1 - n^2 R_0^2)}\, {}^t\bar{e}_{ks}(\mathring{\sigma}) E e_{ks}(\mathring{\sigma}) + \frac{n^2 \sqrt{(1 - R_0^2)}}{\sqrt{(1 - n^2 R_0^2)}}\, {}^t\bar{e}_{kp}(\mathring{\pi}) E e_{kp}(\mathring{\pi}) \\ + \frac{2n(R_0^2 - 2)\sqrt{(1 - R_0^2)}}{\sqrt{(1 - R_0^2)} + \sqrt{(1 - n^2 R_0^2)}} (-i \operatorname{sgn} \xi)\, {}^t\bar{e}_{kp}(\mathring{\pi}) E e_{ks}(\mathring{\sigma}).\end{aligned}$$

Then

$$\int_R {}^t\bar{\tilde{\gamma}}_k(\xi, \rho) E \tilde{\gamma}_k(\xi, \rho)\, d\rho = 4\pi |\xi| \tau(R_0). \tag{3.32}$$

Remark. Since from (3.18) and (3.3),

$$\begin{aligned} {}^t\bar{e}_{ks}(\mathring{\sigma}) E e_{ks}(\mathring{\sigma}) &= R_0^{-4}(4 - 3R_0^2), \\ {}^t\bar{e}_{kp}(\mathring{\pi}) E e_{kp}(\mathring{\pi}) &= n^{-2} R_0^{-4}(4 - 4n^2 R_0^2 + R_0^2), \\ {}^t\bar{e}_{kp}(\mathring{\pi}) E e_{ks}(\mathring{\pi}) &= (i \operatorname{sgn} \xi) 2^{-1} n^{-1} R_0^{-4} \left[(4 - 2n^2 R_0^2 + R_0^2)\sqrt{(1 - R_0^2)} \right. \\ &\qquad \left. + (4 - R_0^2)\sqrt{(1 - n^2 R_0^2)} \right], \end{aligned} \tag{3.33}$$

it follows that

$$\tau(R_0) = \surd(1 - n^2R_0^2) \cdot (4 - 3R_0^2)R_0^{-4} + (1 - n^2R_0^2)^{-1/2}$$
$$\times (1 - R_0^2)(4 - 4n^2R_0^2 + R_0^2)R_0^{-4} + \lambda(R_0)R_0^{-4},$$
$$\lambda(R_0) = \frac{(R_0^2 - 2)\surd(1 - R_0^2)}{\surd(1 - R_0^2) + \surd(1 - n^2R_0^2)}\left[\surd(1 - R_0^2) \cdot (4 + R_0^2 - 2n^2R_0^2)\right.$$
$$\left. + \surd(1 - n^2R_0^2) \cdot (4 - R_0^2)\right]. \tag{3.34}$$

The next lemma is obtained from (3.34) and (3.21) via (3.24) by elementary algebra, but it is not straightforward in the least (in fact, it's the only hard thing in this entire paper).

LEMMA 3.3. *In terms of the number $\alpha(R_0)$ of* (3.21), *the number $\tau(R_0)$ of* (3.34) *has the representation*

$$\tau(R_0) = \alpha(R_0) \cdot \surd(1 - R_0^2). \tag{3.35}$$

The verification of (3.35) uses only (3.24) and elementary algebra, but a head-on assault leads nowhere. We indicate the principal steps. Using (3.24), we obtain the following expressions:

$$\lambda(R_0) = (R_0^2 - 2)\surd(1 - R_0^2)\left[2 + R_0^2 + \tfrac{1}{2}(R_0^2 - 2)^2\right],$$
$$R_0^{-4}(4 - 3R_0^2)\surd(1 - n^2R_0^2) + (1 - R_0^2)(1 - n^2R_0^2)^{-1/2}$$
$$\times (4 - 4n^2R_0^2 + R_0^2)R_0^{-4}$$
$$= R_0^{-4}(1 - n^2R_0^2)^{-1/2}(8 - 6R_0^2 - 8n^2R_0^2 + 7n^2R_0^4 - R_0^4).$$

Thus,

$$\tau(R_0) = R_0^{-4}(1 - n^2R_0^2)^{-1/2}\left[8 - 6R_0^2 - 8n^2R_0^2 + 7n^2R_0^2 - R_0^4\right.$$
$$\left. + (R_0^2 - 2)\surd(1 - R_0^2)\surd(1 - n^2R_0^2) \cdot (4 - R_0^2 + \tfrac{1}{2}R_0^4)\right]$$
$$= R_0^{-4}(1 - n^2R_0^2)^{-1/2}\left[8 - 6R_0^2 - 8n^2R_0^2 + 7n^2R_0^2 - R_0^4\right.$$
$$+ \tfrac{1}{4}(R_0^2 - 2)^3(4 - R_0^2 - \tfrac{1}{2}R_0^4)$$
$$\left. + R_0^4(R_0^2 - 2)\surd(1 - R_0^2) \cdot \surd(1 - n^2R_0^2)\right].$$

Now

$$\tfrac{1}{4}(R_0^2 - 2)^3(4 - R_0^2 - \tfrac{1}{2}R_0^4) = -8^{-1}(R_0^2 - 2)^4(4 + R_0^2)$$
$$= -2(1 - R_0^2)(1 - n^2R_0^2)(4 + R_0^2),$$

and substituting this into the preceding expression for $\tau(R_0)$ and simplifying gives the result.

Having made our mark in algebraic number theory, the rest is easy. We define

$$l_{\pm}(\xi; f) = (4\pi)^{-1}\alpha(R_0)^{-1}\sqrt{(1 - R_0^2)} \cdot \chi_{\pm}(\xi) \times \int_{R_+^2} e^{-i\xi y_1}\bar{\gamma}(\xi, y_2) Ef(y)\, dy \tag{3.36}$$

$$L_{\pm}(x_1 \pm i\alpha x_2; f) = \int_{R_{\pm}} e^{i\xi(x_1 \pm i\alpha x_2)}|\xi| l_{\pm}(\xi; f)\, d\xi, \qquad \alpha \in \mathbb{R}_+, \tag{3.37}$$

$$\begin{aligned} \sigma &= (\sigma_1, \sigma_2) = R_0^{-1}\left(1, i\left(1 - R_0^2\right)^{1/2}\right), \\ \pi &= (\pi_1, \pi_2) = (nR_0)^{-1}\left(1, i\left(1 - n^2R_0^2\right)^{1/2}\right), \\ 1 &= \sigma_1^2 + \sigma_2^2 = \pi_1^2 + \pi_2^2. \end{aligned} \tag{3.38}$$

From (3.31), (3.32), and (3.35) $\Pi_\sigma^k f = (\Pi_\sigma^k)^+ f + (\Pi_\sigma^k)^- f$, where

$$\begin{aligned} (\Pi_k^\sigma)^+ f(x) &= \frac{R_0^2 - 2}{\sqrt{(1 - R_0^2)}} e_{ks}(\sigma) L_+\left(x_1 + ix_2\sqrt{(1 - R_0^2)}\right) \\ &\quad + 2nie_{kp}(\pi) L_+\left(x_1 + ix_2\sqrt{(1 - n^2R_0^2)}\right), \\ (\Pi_k^\sigma)^- f(x) &= \frac{R_0^2 - 2}{\sqrt{(1 - R_0^2)}} e_{ks}(-\bar{\sigma}) L_-\left(x_1 - ix_2\sqrt{(1 - R_0^2)}\right) \\ &\quad - 2nie_{kp}(-\bar{\pi}) L_-\left(x_1 - ix_2\sqrt{(1 - n^2R_0^2)}\right). \end{aligned} \tag{3.39}$$

We observe that it follows from (3.36) and (3.2) by the Cauchy–Schwartz inequality, that $|\xi| l_{\pm}(\xi; f) \in L_2(R, C)$.

THEOREM 3.4. *With* $(\Pi_k^\sigma)^{\pm} f$ *of* (3.39), *any Rayleigh* (*surface*) *wave has the form* ($k = \pm 1$),

$$U(t)\Pi_k^\sigma f(x) = (\Pi_k^\sigma)^+ f\left(x_1 + kt\mu^{1/2}R_0, x_2\right) + (\Pi_k^\sigma)^- f\left(x_1 - kt\mu^{1/2}R_0, x_2\right). \tag{3.40}$$

Conversely, for any $l_{\pm}(\xi) \equiv \chi_{\pm}(\xi) l_{\pm}(\xi) \in L_2(R_{\pm}, C)$ *such that* $|\xi| l_{\pm}(\xi) \in$

$L_2(R_\pm, C)$, let

$$L_\pm(x_1 \pm i\alpha x_2) = \int_{R_\pm} e^{i\xi(x_1 \pm i\alpha x_2)} |\xi| l_\pm(\xi)\, d\xi, \quad 0 < \alpha \in R, \tag{3.41}$$

and define

$$\begin{aligned} f_+(x) &= \frac{R_0^2 - 2}{\sqrt{(1 - R_0^2)}} e_{ks}(\sigma) L_+\big(x_1 + ix_2\sqrt{(1 - R_0^2)}\big) \\ &\quad + 2\, ni\, e_{kp}(\pi) L_+\big(x_1 + ix_2\sqrt{(1 - n^2R_0^2)}\big), \\ f_-(x) &= \frac{R_0^2 - 2}{\sqrt{(1 - R_0^2)}} e_{ks}(-\bar\sigma) L_-\big(x_1 - ix_2\sqrt{(1 - R_0^2)}\big) \\ &\quad - 2\, ni\, e_{kp}(-\bar\pi) L_-\big(x_1 - ix_2\sqrt{(1 - n^2R_0^2)}\big), \\ f &= f_+ + f_-, \end{aligned} \tag{3.42}$$

then $f \in \mathcal{H}$ *and* $(\Pi_k^\sigma)^\pm f_\pm = f_\pm$, $(\Pi_k^\sigma)^\pm f_\mp = 0$, $\Pi_k^\sigma f = f$. *If* $|\xi|^{3/2} l_\pm(\xi) \in L_2(R_\pm, C)$, *then f is furthermore in* $\mathfrak{D}(\mathcal{Q})$. *Finally, it is clear from* (3.39) *that no nonzero data in* $\Pi_k^\sigma \mathcal{H}$ *are compactly supported, and, as in Section* 1, *data in* $\Pi_k^\sigma \mathcal{H}$ *need not decay exponentially away from* $x_2 = 0$.

Remark. In (3.40) each term g on the right individually satisfies (3.1), (3.7), (3.10) and the one-dimensional, scalar wave equation $(\partial_t^2 - \mu R_0^2 \partial_1^2) I_5 g = 0$. It is evident from (3.40) and (3.39) that the structure of Rayleigh waves as vector fields is trivial. Note also that the second assertion together with (3.39) gives a direct verification that $(\Pi_k^\sigma)^2 = \Pi_k^\sigma$.

Proof of the theorem. The first assertion follows immediately from Theorem 3.1. We prove the converse assertion. Note that from (3.36), (3.37),

$$\begin{aligned} \int e^{-i\xi y_1} L_\pm(y_1 \pm i\alpha y_2)\, dy_1 &= e^{\mp\alpha\xi y_2} (2\pi)^{1/2} \Phi_1 L_\pm(\xi) \\ &= 2\pi \chi_\pm(\xi) e^{\mp\alpha\xi y_2} |\xi| l_\pm(\xi), \end{aligned} \tag{3.43}$$

so that $l_\pm(\xi; f_\mp) = 0$, and hence $(\Pi_k^\sigma)^\pm f_\mp = 0$. It remains to show that $(\Pi_k^\sigma)^\pm f_\pm = f_\pm$, i.e., that $l_\pm(\xi; f) = l_\pm(\xi)$. We show that $l_+(\xi; f_+) = l_+(\xi)$; the case of l_- is the same. To abbreviate notation, we set $\alpha_1 = \sqrt{(1 - R_0^2)}$, $\alpha_2 = \sqrt{(1 - n^2R_0^2)}$; then, writing $l_+(\xi) \equiv \chi_+(\xi) l_+(\xi)$, from (3.20), (3.24), (3.33), (3.36), (3.38), (3.42), and (3.43), we have

$$\begin{aligned} l_+(\xi; f_+) &= 2^{-1}\alpha(R_0)^{-1}\sqrt{(1 - R_0^2)} \times \\ &\quad \times \xi l_+(\xi) \int_0^\infty \left\{ e^{-\xi y_2 \alpha_1} \frac{R_0^2 - 2}{\sqrt{(1 - R_0^2)}}\, {}^t\bar{e}_{ks}(\sigma) E - e^{-\xi y_2 \alpha_2}\, {}^t\bar{e}_{kp}(\pi) E \right\} \end{aligned}$$

$$\times \left\{ \frac{R_0^2 - 2}{\surd(1 - R_0^2)} e^{-\alpha_1 \xi y_2} e_{ks}(\sigma) + 2\,\mathrm{nie}^{-\alpha_2 \xi y_2} e_{kp}(\pi) \right\} dy_2$$

$$= \alpha(R_0)^{-1} \surd(1 - R_0^2) l_+(\xi) \left\{ \frac{\surd(1 - n^2 R_0^2)}{1 - R_0^2}\, {}^t\bar{e}_{ks}(\sigma) E e_{ks}(\sigma) \right.$$

$$+ \frac{n^2}{\surd(1 - n^2 R_0^2)}\, {}^t\bar{e}_{kp}(\pi) E e_{kp}(\pi)$$

$$\left. - \frac{2n(R_0^2 - 2)}{\surd(1 - R_0^2)[\surd(1 - R_0^2) + \surd(1 - n^2 R_0^2)]} \operatorname{Im} {}^t\bar{e}_{ks}(\sigma) E e_{kp}(\pi) \right\}$$

$$= \alpha(R_0)^{-1} (1 - R_0^2)^{-1/2} l_+(\xi) \{ \surd(1 - n^2 R_0^2)$$

$$\cdot (4 - 3R_0^2) R_0^{-4} + (1 - n^2 R_0^2)^{-1/2} (1 - R_0^2)$$

$$\times (4 - 4n^2 R_0^2 + R_0^2) R_0^{-4} + \lambda(R_0) R_0^{-4} \}$$

$$= \alpha(R_0)^{-1} (1 - R_0^2)^{-1/2} l_+(\xi) \tau(R_0) = l_+(\xi)$$

by (3.35). The assertion regarding $f \in \mathfrak{D}(\mathfrak{A})$ follows in the same way on the basis of Theorem 3.1.

4. Concluding Remarks

We remark that in problems with layered media the so-called trapped or guided modes have a structure similar to that of the surface waves discussed here. In particular, they propagate according to scalar wave equations. Such waves are of considerable importance in fiber optics, geophysical prospecting, etc., and a good deal of work remains to be done in studying them along the lines set forth here.

The present approach has been developed over the past five years from an initial purpose of constructing a system of examples sufficiently manageable to indicate how substantial information might be derived from formalisms of scattering-theory type. For example, although the avowed purpose of generalized-eigenfunction expansions is to obtain more information about solutions than afforded by the spectral theorem, many papers in this area (including our own), after many pages of impenetrable analysis, leave the matter with the conclusion that solutions consist of superpositions of "distorted plane waves." While this deduction may be received as quite a revelation in the sophisticated circles for which it is intended, from the standpoint of applying mathematics to useful ends it is simply a word game. We believe to have demonstrated above that solid information about solutions can be obtained from spectral theory if one has the courage to dig it out. This we consider to be the principal merit of the paper.

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